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TRANSIENT FREE CONVECTION ON AN ISOTHERMAL VERTICAL FLAT PLATE

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NOMENCLATURE

g, acceleration due to gravity;
M, N, number of mesh points in Y and X directions respectively;
Pr, Prandtl number $(= \nu/\alpha)$;
R, a "Reynolds" number $[= 2/(U\Delta X)]$;
t, dimensionless time $[= \text{time}\ (g\beta_1 \Delta T)^{2/3}/\nu^{1/3}]$;
Δt, dimensionless time step;
T, dimensionless temperature $[= (\text{Temperature} - T_i)/\Delta T]$;
T_i, initial temperature of fluid;
T_w, temperature of plate;
ΔT, $T_w - T_i$;
u, v, velocities in the x and y directions respectively;
U, dimensionless velocity in x direction $[= u/(\nu g\beta_1 \Delta T)^{1/3}]$;
V, dimensionless velocity in y direction $[= v/(\nu g\beta_1 \Delta T)^{1/3}]$;
x, y, distances along the plate from the leading edge and normal to the plate respectively;
X, dimensionless distance along the plate from the leading edge $[= x(g\beta_1 \Delta T/\nu^2)^{1/3}]$;
Y, dimensionless distance normal to the plate $[= y(g\beta_1 \Delta T/\nu 2)^{1/3}]$;
X_{max}, Y_{max}, maximum values of X and Y taken respectively;
$\Delta X, \Delta Y$, dimensionless finite difference step lengths in the X and Y directions respectively.

Greek symbols

α, thermal diffusivity of fluid;
β_1, volumetric coefficient of thermal expansion;
β, dimensionless heat-transfer coefficient $[= -X^{1/4}(\partial T/\partial Y)_{Y=0}]$;
ν, kinematic viscosity of fluid;
τ, a similarity variable $(= t/X^{1/2})$;
τ_c, the value of τ up to which the analytical solution for small values of τ is valid.

INTRODUCTION

THE PROBLEM of a semi-infinite vertical plate which is situated in an infinite fluid that is initially cold and at rest and then impulsively heated has been investigated by several authors. The fluid has been assumed incompressible except for a temperature dependent buoyant effect in the momentum equation and the boundary-layer assumptions have been made. By defining an appropriate dimensionless temperature T, distances X along the plate from the leading edge and Y perpendicular to the plate then the governing equations are, see Hellums and Churchill [1],

Momentum:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = T + \frac{\partial^2 U}{\partial Y^2}, \tag{1}$$

Energy:

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial Y^2}, \tag{2}$$

Continuity:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{3}$$

Here U and V are the dimensionless velocities in the X and Y directions respectively, t is the dimensionless time and Pr the Prandtl number of the fluid. Equations (1)–(3) have to be solved subject to the initial and boundary conditions,

$$\begin{aligned} X = 0&: \quad U = V = T = 0, \\ Y = 0&: \quad U = V = 0,\ T = 1, \\ Y \to \infty&: \quad U, V, T \to 0, \\ t = 0&: \quad U = V = T = 0. \end{aligned} \tag{4}$$

Equations (1)–(3) have been solved using a numerical technique as described by Hellums and Churchill [1] and Carnaham *et al.* [2]. Callahan and Marner [3] have recently solved this problem by the same technique with the addition of the effects of mass transfer. Brown and Riley [4] reduced equations (1)–(3) to similarity form and obtained analytical results which are valid for small and large values of the similarity variable $\tau = t/X^{1/2}$. Further, they predict a time up to which their unsteady solution for small time is applicable, say τ_c. On using standard methods of dealing with these types of similarity equations, namely Hall [5], Dennis [6] and Watkins [7] no solutions which satisfactorily match the analytical solutions for small and large times have yet been obtained. The numerical results presented in [1–3] all show a breakaway from the unsteady solution before τ_c. It therefore seems appropriate to investigate more thoroughly the methods described in [1–3] to see if results consistent with the theory can be obtained. If not, an explanation must be sought and there seems little point in adding in other effects into equations (1)–(3) until this point has been cleared.

RESULTS AND CONCLUSIONS

As taken by Hellums and Churchill [1] we consider the height of the plate to be $X_{max}(= 100)$ and the maximum value of Y to be $Y_{max}(= 25)$ as corresponding to infinity and for simplicity a constant mesh size is used in the X and Y directions, namely $\Delta X = X_{max}/N$ and $\Delta Y = Y_{max}/M$ respectively, where M and N are integers. Derivatives in equations (1) and (2) are written in central differences for $\partial^2/\partial Y^2$ terms, backward for the $\partial/\partial X$ terms and forward for the $\partial/\partial Y$ and $\partial/\partial t$ terms. For $Pr = 0.733$ the results obtained here are identical with those presented in [1] and [2] for the mesh sizes used, namely $N = M = 10$ and $N = M = 40$. Brown and Riley's [4] results are presented for $Pr = 1$ and

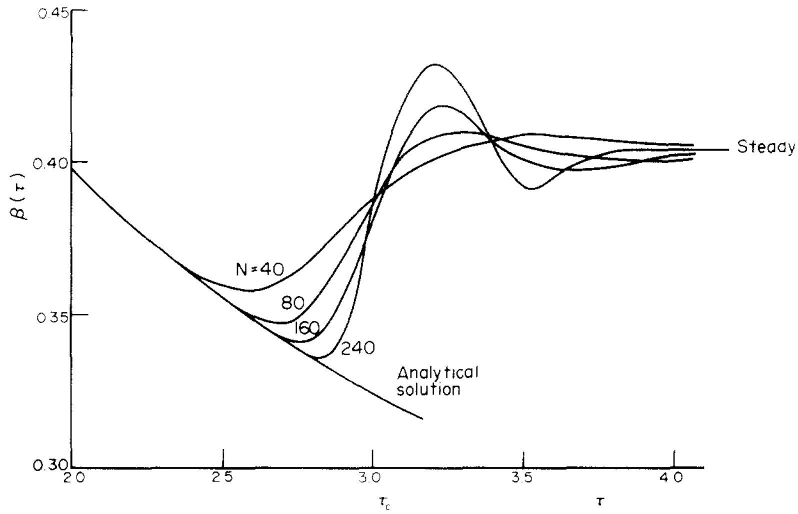

FIG. 1. Variation of β with τ for various values of N.

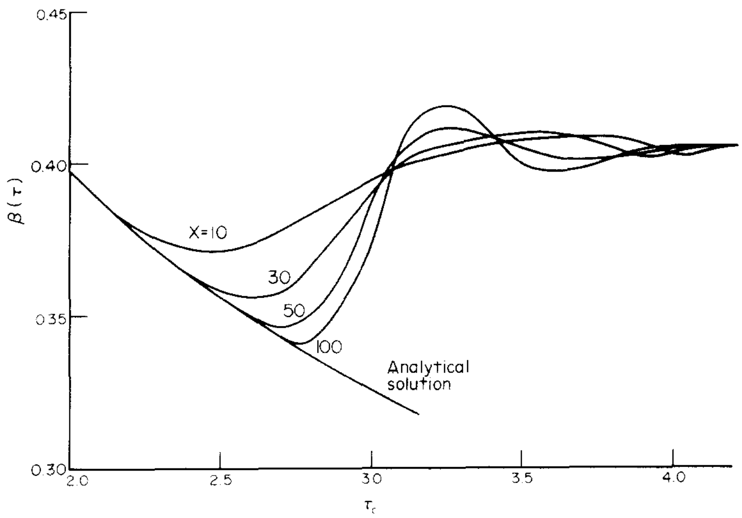

FIG. 2. Variation of β with τ at various values of X.

therefore attention was focussed on this value of the Prandtl number. Figure 1 shows the variation of the dimensionless heat-transfer parameter $\beta[= -X^{1/4}(\partial T/\partial Y)_{Y=0}]$ as a function of τ evaluated at X_{max} for $M = 40$ and $N = 40, 80, 160$ and 240. The exact theoretical variation as determined by Brown and Riley [4] for small values of τ is also shown, the termination point of the curve is the limit of validity of their solution, τ_c. It is seen that as $\Delta X \to 0$ the results tend to remain on the correct analytical solution for larger values of τ but once the solution breaks away from this solution the change in β with τ is very rapid and overshoots and oscillations occur. The smaller ΔX becomes then (i) the larger is the maximum gradient of β with τ, and (ii) the quicker is the transition from the unsteady to the steady solution. The variation of β with τ using a fixed value of ΔX and several different values of ΔY showed very little difference.

Figure 2 shows the variation of β with τ computed at several different stations X along the plate and $M = 40$ and $N = 160$ and the analytical unsteady solution. Similar results apply for other values of M and N. This shows that the numerical results obtained do not exhibit the similarity property. For a fixed value of M and N as the value of X increases the numerical solution agrees longer with the analytical solution and comments (i) and (ii) above also apply to the taking of larger values of X.

An investigation of the consistency of the finite difference equations obtained from the discretization of equations (1)–(3) shows that they are consistent as Δt, ΔX and $\Delta Y \to 0$. It is easily seen that the finite difference equations representing equations (1) and (2) when interpreted back into partial differential form, using *central* differences throughout, are

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = T + \frac{\partial^2 U}{\partial Y^2} + \left[\Delta t\left(\frac{\partial T}{\partial t} - \frac{1}{2}\frac{\partial U}{\partial t}\right) + \frac{U\Delta X}{2}\frac{\partial^2 U}{\partial X^2} - \frac{V\Delta Y}{2}\frac{\partial^2 U}{\partial Y^2}\right] + O(\Delta t^2, \Delta X^2, \Delta Y^2), \quad (5)$$

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial Y^2} + \left[-\frac{\Delta t}{2}\frac{\partial T}{\partial t} + \frac{U\Delta X}{2}\frac{\partial^2 U}{\partial X^2} - \frac{V\Delta Y}{2}\frac{\partial^2 T}{\partial Y^2}\right] + O(\Delta t^2, \Delta X^2, \Delta Y^2). \quad (6)$$

As $\Delta t \sim \Delta Y^2$, for stability of the numerical scheme, the terms in Δt are small. The terms $(V\Delta Y/2)(\partial^2 U/\partial Y^2)$ and $(V\Delta Y/2)(\partial^2 T/\partial Y^2)$ add (since V is negative) to the terms

$\partial^2 U/\partial Y^2$ and $\partial^2 T/\partial Y^2$ in equations (5) and (6) respectively but as ΔY and V are small these discretization errors will not have a marked effect on the numerical calculations. This was in fact verified numerically in that the solutions were not too sensitive to the choice of ΔY. The terms $(U\Delta X/2)(\partial^2 U/\partial X^2)$ and $(U\Delta X/2)(\partial^2 T/\partial X^2)$ effectively add, artificially, diffusion in the X direction. Thus if $Pr = 1$ we may consider the coefficients of these second derivatives in X being inversely proportional to a "Reynolds number", R say, i.e.

$$R = \frac{2}{U\Delta X}. \tag{7}$$

With the values of U and ΔX used in the calculations it is found that near the maximum value of U then R is $O(1)$. Obviously near the plate and at the outer edge of the boundary layer U is very small and the value of R is therefore very large. Since it is at the maximum value of U where the disturbance from the leading edge is travelling fastest, which in turn determines the position up to which the unsteady solution is valid, it is important that the detail of the solution should be correct here. Because of this artificial adding of a large amount of diffusion in the X direction in this critical region it is not therefore surprising that the results of this method do not agree with the analytical results.

It should be remembered that a possible mathematical solution to this problem is that for $\tau < \tau_c^*$, some critical value of τ, the unsteady solution holds whereas for $\tau > \tau_c^*$ the steady solution is valid and the solution is discontinuous at $\tau = \tau_c^*$. If this is the only solution to equations (1)–(3) it is not surprising that the results obtained using the methods described in [5–7] may fail to converge. Whereas using the method described in [1–3], and used here, which artificially adds diffusion in the X direction will obviously converge to an answer which in the limit as $\Delta X \to 0$ or $X_{max} \to \infty$ and $\Delta Y \to 0$ will approach the true solution. As observed from Figs. 1 and 2 taking these limits for a fixed ΔY shows a tendency for the transition from the unsteady to the steady solution to be more rapid and maybe the above mathematical solution is the one to which the results are tending.

One may add in a variable mesh in order to reduce the computing time and yet keep good detail near $X = 0$ and $Y = 0$ as was used by Callahan and Marner [3] but the values of $\Delta X/X_{max}$ and $\Delta Y/Y_{max}$ used here were smaller than the finest meshes used in [3]. Although they quote results which agreed to within 7% when the space mesh was halved one can see from the results presented here that this is not a sufficient test to the accuracy of the method.

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USE OF THE MICROTRANSIENT THERMAL DIFFUSIVITY MEASURING TECHNIQUE ON LIQUIDS: IMPLICATIONS ON FLUIDS OF THE "WEAK INFRARED ABSORBING" CLASS

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NOMENCLATURE

- C, temperature [°C];
- m, distance [m];
- C_p, specific heat [J/kg°C];
- α, thermal diffusivity [m²/s];
- k, thermal conductivity [W/m°C];
- ρ, density [kg/m³].

INTRODUCTION

THE PURPOSE of this short communication is to report the results of new work performed with the "microtransient diffusivity measuring system" since that previously reported [1, 2], and to summarize experimental modifications effected to increase measurement accuracy. Specifically, the new work addressed the class of so called "weak IR absorbing" fluids [3, 4]. For this class of fluids, including carbon tetrachloride and toluene, the results support the conclusion that experimental techniques for measuring thermal conductivity may have a systematic error in results reported, if care is not taken to discriminate against the detection of energy transfer by radiation when energy transfer by conduction is the desired process being measured.

The data generated by this technique were for measurement distances significantly smaller than those used by any other existing heat-transfer measurement techniques, and implied for the "small" distances used a larger radiation contribution potential than previously determined for "weak IR absorbing" fluids.

NEW MEASUREMENTS, AND RESULTS FOR LIQUIDS OF THE WEAKLY IR ABSORBING CLASS

By way of summary, Appendix A presents a schematic of the measuring technique developed and already reported, as well as modifications made to the measuring system to increase the precision of the measurement from that originally developed. In the apparatus, as it is currently used, a cylindrical line source (~200 μm long) acts as a short duration transient energy input (~200 μs) within a transparent test fluid. The source, created from a biological laser coupled to an inverted metallurgical microscope, induces shock, radiation, and thermal transients in the test fluid. The

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